AN INSTRUCTOR'S ACTIONS FOR MAINTAINING THE COGNITIVE DEMANDS OF TASKS IN TEACHING MATHEMATICAL INDUCTION

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Mathematical tasks are central to students' learning since they can influence and structure the ways in which students think about mathematics. Carefully selected tasks have potential to broaden students' views of a subject matter and facilitate their mathematical growth. However, research identifies that cognitive demands of tasks may change as the tasks are enacted during instruction. For this reason, it is important to understand what instructors can do to maintain the intended cognitive demands. In this paper, we investigate a teacher's actions for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction. Our findings suggest that the method of quasi-induction (Harel, 2002) may be considered as an example of a productive scaffolding strategy for assisting students in mastering proof by induction.

Keywords: Classroom Discourse, Reasoning and Proof, University Mathematics

Proof by mathematical induction is a technique for proving statements about natural numbers. To prove that proposition $P(n)$ holds for any natural number n, one needs to check that (1) $P(1)$ is true (the base case) and (2) if $P(k)$ is true for some fixed but arbitrary natural number k, then $P(k+1)$ is also true (inductive implication). The principle of mathematical induction poses conceptual difficulties to college students (Dubinsky, 1991; Harel, 2002; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007).

Carefully selected tasks can help students overcome cognitive obstacles associated with proof by mathematical induction. Mathematical tasks play a crucial role in students' learning. They can shape students' conceptions about the subject. Furthermore, they offer an opportunity for teachers to lower their authority in the classroom, in turn allowing students to create mathematics for themselves. However, one must be able to strike a balance when determining the appropriate difficulty of task for a student. When tasks do not significantly challenge the student, they may become routine or discourage creativity. In contrast, if a task is too difficult, students may make insufficient progress toward the intended mathematical goal.

The cognitive demand of a task represents its level of difficulty (Stein, Grover, & Henningsen, 1996). The cognitive demands of tasks for K-12 students have been well documented (Spears and Chávez, 2014; Bieda, 2010; Henningsen and Stein, 1997). However, to our knowledge, cognitive demands of have not been extensively explored at the undergraduate level. This study aims to contribute to the research on cognitive demands of tasks by considering problems an instructor used in teaching proof by mathematical induction. Specifically, the purpose of this case study is to investigate a teacher's actions during the enactment of high-level tasks. Results address the following research question: *what are the teachers' actions for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction?*

Theoretical Framework

The present study is guided by the Mathematical Tasks Framework (Stein et al., 1996). This model describes the evolution of a task through three phases of classroom: as written in instructional materials, as set up by a teacher in the classroom, and as implemented by the students. Ultimately, a mathematical task should lead to student learning.

For the purposes of this study, we distinguish between *planned* and *enacted* mathematical instruction. Planned instruction refers to how teachers plan a mathematical task and how they pose it in the classroom. Enacted instruction is the actual teaching that occurs, including the active roles of both teachers and students (Remillard, 2005). The Mathematical Tasks Framework represents planned instruction by the first two phases and enacted by the third one.

The framework further specifies two dimensions of tasks, task features and cognitive demands, that may affect the transition between phases. This study is centered around cognitive demand. Cognitive demand refers to the variation in the kind of thinking processes required of students while engaging with the tasks. According to Stein et al., (2009), there are four levels of cognitive demands: a) memorization, b) procedures without connections, c) procedures with connections and d) doing mathematics. The first two levels are traditionally considered low-level demands, while the latter refer to high-level demands.

The enactment of mathematical tasks of high cognitive demands allows students to develop sense-making and reasoning skills, critique their peers' solutions, formulate examples and counterexamples, create viable justifications, and properly communicate their reasoning. Therefore, implementation of high-level tasks may be beneficial for students' mathematical growth. Planning the implementation of a task is crucial to students' learning that occurs around this task. However, research identifies that teachers have difficulty enacting high-level tasks even if they were planned as such (Boston & Smith, 2009). Teachers may either maintain the high cognitive level or they may lower it to make tasks more accessible for students. Stein and Smith (1998) suggest a list of factors associated with the maintenance of high-level cognitive demands. These factors include teachers giving sufficient time, making conceptual connections, pressing for justifications, and scaffolding student thinking and reasoning.

The term scaffolding has been used in various contexts. Anderson (1989) highlighted the importance of the Vygotskian notion of scaffolding in supporting students' high-level thinking processes. Henderson and Stein (1997) define scaffolding as a teacher's assistance in response to a student's struggle with a task. This assistance enables the student to complete the task alone, but does not reduce the cognitive demands of the task. William and Baxter (1996) separate the constructs of analytic and social scaffolding. Social scaffolding refers to the scaffolding of social norms; analytic scaffolding is the "scaffolding of mathematical ideas for students" (p. 24). Speer and Wagner (2009) consider analytic scaffolding as guiding students "further toward the desired mathematical goal(s) by using selected student contributions" (p. 536). For the purposes of our data analysis, we put these ideas together and define the construct of scaffolding as a *teacher's pedagogical strategies or actions toward the desired mathematical goals in response to or in anticipation of students' struggles.* Furthermore, we introduce the term *productive scaffolding* to refer to *scaffolding that maintains the cognitive demands of the task.*

Data and Methods

This study used one white male instructor's materials and three episodes of teaching proof by mathematical induction at a large public research university in the southeastern United States. The course is a junior-level course designed to teach mathematics majors typical mathematical proof techniques. The data used are part of a larger project studying cognitive components of proof by mathematical induction. For this project, research-based instruction was developed and implemented. Teaching episodes were video and audio recorded and transcribed by the authors.

We analyzed the instructor's lecture notes as written prior to instruction. All mathematical tasks were categorized using the Tasks Analysis Guide (TAG) (Stein et al., 2000) with respect to Stein et al.'s (1996) levels of cognitive demands. Once a consensus was reached between us, we went through two phases of video analysis and coding. During the first round of video analysis, we classified each mathematical task's level of cognitive demand as it was presented in the classroom. We then coded the teacher's actions for instances of Stein et al.'s (2009) factors for maintaining cognitive demands, using Vygotsky's notion of scaffolding. After identifying the factors, we returned to our data to closely study the instances of scaffolding using our definition.

Results and Discussion

During the observed teaching episodes, the instructor used PowerPoint slides with three tasks displayed on three big screens around the room (Figure 1).

1. For each of the following parts, decide whether the given information is enough to conclude that the following claim is true.

   Claim: $P(n)$ is true for all $n \in Z^{+¿¿}$.

   If the given information is not enough, offer a brief explanation on why (perhaps listing a value of n for which $P(n)$, is not known to be true).

   a) $P(1)$is true and there is an integer $k \geq 1$,such that $P(k) \rightarrow P(k+1)$.
   b) $P(1)$is true and for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
   c) For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
   d) $P(1)$ is true and for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$.
2. a) Prove that for all natural numbers n, 3 divides $8^n - 5^n$.

   b) Prove that if 3 divides $8^6 - 5^6$, then 3 divides $8^7 - 5^7$.
3. Prove that for all natural numbers n, $2+2^2+2^3+\ldots+2^n = 2^{n+1} - 2$.

Figure 1: Tasks

When presenting the tasks to the class, the instructor provided students with little preliminary explanation. He typically displayed the tasks on the screens and encouraged students to work in small groups. For this reason, we can claim that these tasks were set up by the teacher in the same way that they were presented in the instructional materials. In the following discussion, we will refer to the tasks of first or second phase (Stein et al., 1996) as planned.

We further used TAG as an instrument to analyze the cognitive demands of tasks. All the tasks exhibited key attributes of "Doing Mathematics" tasks (see Stein et al., 2000). The problems required complex and non-algorithmic thinking and considerable cognitive effort. They also encouraged students to explore and understand the nature of mathematical concepts and to access relevant knowledge and experience.

The first round of analysis revealed the presence of most of the factors associated with maintaining high-level cognitive demands of tasks (Henningsen & Stein, 1997, Boston & Smith, 2009). First, the instructor seemed to allot an appropriate amount of time for the students to engage with the tasks through small-group discussion. Second, students received formal instruction on logical implication before they were introduced to the principle of mathematical induction. Given that logical implication is an important part of each task, we can argue that the tasks build on students' prior knowledge. Third, the instructor constantly questioned students by asking them to rephrase and justify their reasoning. He also frequently made conceptual connections between the tasks and students' solutions and modeled high-level performance through presenting counterexamples to students' erroneous claims.

Scaffolding

Avital and Libeskind (1978) introduced the method of "naïve induction" to assist students in overcoming their bewilderment of the transition from the base case to the inductive step. Naïve induction has been elaborated by Harel (2002) and labeled as "quasi-induction." Prior research has identified quasi-induction as a fruitful instructional approach (Harel, 2002; Cusi & Malara, 2008). This method engages students in repeated application of the inductive implication $P(k) \to P(k+1)$ for beginning values of k, reinforcing the logical reasoning that is essential in proof by mathematical induction. More specifically, students first establish that $P(1)$ is true. Then, they create a chain of logical implications $P(1) \to P(2), P(2) \to P(3), P(3) \to P(4)$,and so on. After considering these first few implications, students can then infer that the process must continue until eventually $P(n)$ is shown to be true.

The idea of quasi-induction was built into the overall observed instruction. In anticipation of students' struggle with formal proof, Tasks 1 and 2 were designed to engage students in quasi-inductive reasoning. Quasi-induction was first explicitly introduced by one of the students in the discussion of Task 1c who said, "1 works, so $P(1+1)$ works, so 2 works. And you can plug 2 back in for k and the logic repeats itself." In response to the student's reasoning, the instructor discussed mathematical rigor of quasi-induction, but accepted the suggested solution. Furthermore, during the group work on Task 3, students in one of the groups were not engaged with the task. The instructor suggested they use quasi-induction: "Sometimes it helps to do – just try a bunch of cases, just to get a feel of what's going on." This prompt allowed the task to still have the key attributes of Doing Mathematics while making it more accessible for the students.

Task 2a was introduced at the very beginning of the first class. One of the students suggested using the binomial expansion to prove the statement. The teacher acknowledged this idea but encouraged the students "to practice something inductive." The instructor anticipated students' struggle with formal solution using proof by mathematical induction. For this reason, after the students went through Tasks 1a-1d, he presented an "easier" Task 2b. The use of Task 2b did not reduce the cognitive demand of Task 2a. However, the students were given means to generalize the proof of $P(k) \to P(k+1)$ from a particular case of logical implication $P(5) \to P(6)$. Therefore, we consider the idea of generalization from a particular case as another example of what we call productive scaffolding.

Conclusion

This study provides an example of instructor's strategies for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction. The actions discussed above may inform instructors in preparation for and implementation of teaching mathematical induction. Our findings are consistent with the factors suggested by the extant literature (Henningsen & Stein, 1997, Boston & Smith, 2009 Stein & Smith, 1998). Namely, the teacher actively built upon students' prior knowledge, constantly asked students to explain their reasoning, and purposefully facilitated conceptual connections. We also report that scaffolding plays a central role in teaching proofs. In the context of proof by mathematical induction, the method of quasi-induction is suggested to be an example of what we call a productive scaffolding.

Although formal mathematical induction may be considered as a generalization of quasi-induction, there is still a cognitive gap between the two, which the students are not always able to bridge. Harel (2002) described this gap as a difference in perception of the inference $P(n) \to P(n+1)$. Future research must elucidate scaffolds as students attempt to bridge the gap.

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