The Role of Gestures in Teaching and Learning Proof by Mathematical Induction

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When talking about mathematics, teachers and learners actively use hand gestures to support their speech as well as to describe ideas that are not expressed verbally. In this study, I investigate the gestures that were utilized by an instructor and his students during a teaching episode on proof by mathematical induction. Alibali and Nathan's (2012) typology of gestures are employed to code the observed gestures. The study reveals that the use of gestures plays an integral role in teaching and learning induction. I show that pointing gestures helped to reduce ambiguity in classroom discussion, representational gestures were useful in describing specific subcomponents of induction, and, finally, metaphoric gestures were independently introduced by a teacher and students to describe the nature of proof by mathematical induction.

Keywords: gestures, embodied cognition, proofs, mathematical induction.

Literature Review

A number of scholars argue that gestures convey meaning and should be considered as an important part of communication (Lakoff & Nunes, 2000; Goldin-Meadow, 2003; Radford, 2003). People use gestures not only to support their speech, but also to describe ideas that are not expressed verbally, even without realizing it. Goldin-Meadow (1997) characterized gesture as a "window to the mind". For this reason, gestures have become the focus of attention for many psychologists, neuroscientists and educators; a comprehensive analysis of gestures may help to understand the way of human's thinking.

The investigation of gestures in mathematics takes place within a philosophical perspective that frames cognition as an embodied phenomenon. Wilson (2002) described the nature of embodied cognition by noting that "the mind should be understood in the context of its relationship to a physical body that interacts with the world" (p. 625). As a result, human cognition has deep roots in sensorimotor processing. Alibali and Nathan (2012) argued that the cognition is embodied in two senses – being based in perception and action, and being grounded in physical environment.

Gestures play an important role in the learning and development of children (Piaget, 1959). Research confirms that gesturing facilitates students' learning of mathematics in different contexts, such as learning to count (Alibali & diRusso, 1999), symmetry (Valenzeno, Alibali & Klatzky, 2003), equivalence (Singer & Goldin-Meadow, 2005), ratio and proportion (Abrahamson, 2003), motion and graphing (Nemirovsky, Tierney & Wright, 1998). Students often benefit from their gesturing while engaging in challenging, abstract mathematics since the hand gestures help to convey meaning without requiring an overwhelming amount of cognitive resources (Cook, Duffy & Fenn, 2013; Goldin-Meadow, Nusbaum, Kelly & Wagner, 2001). Gestures may help to represent concepts students find difficult to remember and allow learners to coordinate abstract mathematical relationships when processing multiple tasks (Alibali & Nathan, 2012; Ping & Goldin-Meadow, 2010). Gestures, therefore, are key elements in students' processes for knowledge objectification. Gestures help learners to put together different pieces of information and understand conceptually difficult mathematical objects.

Gestures are also actively used by teachers (Flevares & Perry, 2001; Neill, 1991). Literature suggests that students may benefit from their teachers' gestures. Research confirms that learners can detect conceptual information expressed in gestures, and that information that teachers express in gestures facilitates learning (Kelly & Church, 1998). Many studies have shown improvement in students' performance on a posttest after lessons taught by a teacher actively using gestures, compared to lessons that did not contain gestures (Church, Ayman-Nolley, & Mahootian, 2004).

Understanding the connections between different mathematical objects plays an important role in learning mathematics, and guiding students to make those connections is an important aspect of instruction. Several studies aimed to identify how teachers communicate connections among ideas in instruction in classroom settings (for example, Richland, Zur, & Holyoak, 2007). Their findings confirm that teachers often employ gestures to refer to the ideas being connected.

Proof by mathematical induction is known to be conceptually difficult for undergraduate students (Harel, 2002; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007). This method is used to prove that the statement P(n) holds for any natural number n. To prove P(n) by mathematical induction, one must check two assumptions: (a) the validity of P(1) (the base case), and (b) if the statement P(k) is true for some natural number k, than it is also true for P(k+1) (inductive implication). The purpose of this case study is to investigate the role of gesturing in teaching and learning proof by mathematical induction. More specifically, the study is guided by the following research questions:

- *How does the instructor use gestures in teaching proof by mathematical induction?*
- *How do students use gestures in learning proof by mathematical induction?*

Theoretical Framework

I draw on Alibali and Nathan's (2012) typology of gestures manifesting embodied cognition. *Pointing gestures* reflect the grounding of cognition in the physical environment. This type of gestures is the most commonly used in mathematics (Alibali, Nathan & Fujimori, 2011). Pointing gestures are usually used to indicate objects, location, inscriptions or students.

Representational gestures convey simulations of action and perception and depict semantic content, literally or metaphorically, with the help of handshape or motion trajectory. According to Alibali and Nathan (2012), actions and perceptions are intimately linked: "when humans perceive objects, they automatically activate actions appropriate for manipulating or interacting with those objects" (p. 254). From this perspective, actions and perceptions are similar and it does not really matter what exactly, action or perception, a gesture represents.

Metaphoric gestures are a subset of representational gestures and reflect conceptual metaphors that are grounded in the body. These metaphors transmit understanding and perceptions of the reality. The conceptual metaphors that underlie mathematical ideas are often expressed in the gestures that humans produce when speaking about these ideas. Thus, metaphoric gestures provide evidence for psychological underpinnings of mathematical concepts.

Data Source

The study utilized a 40-minute teaching episode on proof by mathematical induction in a large research university in the southeastern United States. The course is a junior-level mathematics course designed to teach mathematics major students typical mathematical proof techniques. The participants were a white male mathematics professor and 20 students, who agreed to participate in the study. The video-data were part of a larger project studying teacher-students' interactions during instruction on mathematical induction. The videotapes were completely transcribed by the author, and the segments of classroom discourse when the teacher or a student gestured were marked and recorded on the transcripts. These segments were then analyzed qualitatively for the types of gestures that were used and the role the gestures played in facilitating the construction of shared meaning between the teacher and students.

Methods and Procedures

Before documenting any of the teacher's gestures, the author watched the videotapes and read the transcripts a few times. Then, using classroom videotapes, all of the teacher's and students' gestures were time indexed for starting time and duration, and classified in accordance with Alibali and Nathan's (2012) framework. Given that gestures occur very quickly, I re-watched the videotapes to assure that all the gestures and accompanying speech were documented. The gestural episodes that did not refer to mathematical instruction or mathematical conversation were omitted from the analysis.

Results and Discussion

In total, there were 132 instances of teacher gesturing and 15 incidents of student gestures. The huge difference between the number of teacher and student gestures may be explained by the fact that the observed teaching episode was an introduction to proof by mathematical induction, and, therefore, a considerable part of classroom discourse was held by the instructor. I report on the use of gesturing in accordance with Alibali and Nathan's (2012) typology.

Pointing gestures

Alibali and Nathan (2012) argue that pointing gestures reveal indexing of speech to the environment. In this subsection, I present how the instructor and students utilized pointing gestures to index mathematical ideas in proof by induction to the physical world.

The data confirm Alibali, Nathan and Fujimori's (2011) claim that pointing gestures constitute a majority of gesturing: 70% of the teacher's and 54% of the students' gestures may be characterized as pointing gestures. All the instances of pointing gestures were accompanied by an utterance "this," "that," or "it," or by addressing a mathematical symbol written on the board. The use of pointing gestures not only supported verbal communication between teacher and students, but also helped to circumvent the ambiguity of the words "this" and "it" in a classroom discourse.

Figure 1. Example of a pointing gesture.

One of the key aspects of teaching proof by mathematical induction is helping students distinguish between the truth for proposition P(k) versus the implication P(k) → P(k+1). During the observed teaching episode, the instructor purposefully pointed to either P(k) or to the sign of implication written on the board to indicate which one he referred to (Figure 1). However, students' responses show that they did struggle to understand the difference:

Teacher: What integers do we know that the proposition works for? … I know, **it** works for 1. How do I know, **it** works for 2?
Student: I'm saying that there is an integer k ≥ 1. So, we can say that k is equal to 1. So, we only know that k works for 1 and 2.
Teacher: How do we know that **it** works for 2, that's my question… How do we know that **it** works for 3?
Student: Because the P(k), k is equal to 1.

Here, the student conflated the truth for proposition P(1) with the truth for implication P(1) → P(2). In this example, the teacher's word "it" was not supported with the corresponding pointing gesture and, consequently, could lead to confusion among the students.

Representational gestures

As it was noted by Alibali and Nathan (2012), "representational gestures simulate real-world objects that ground or give meaning to mathematical ideas" (p. 264). The present subsection describes how this type of gesture was employed in the classroom.

Any statement that may be proved by mathematical induction contains quantifiers. In order to prove that implication P(k) → P(k+1) is valid *for any* k, one should first assume that P(k) is true *for some arbitrary* natural k. The difference between these two quantifiers is subtle but critical. That is why it is crucial for students to understand the role of quantifiers in proof by induction.

The instructor and students used representational gestures in multiple ways. However, a large majority of teacher's gestures were aimed to represent quantifiers "for all", "for any" and "for some". The instructor repeatedly relied on the hand gestures that appeared like a fountain (spreading out and up) when he referred to the goal of the task – to prove the validity of proposition P(k) *for all* positive integers:

Teacher: Ok, let's put your ideas together. We know that P(1) is true. And there is some k ≥ 1 s.t. the proposition is true for k implies the proposition is true for k+1. Is that enough to know that the proposition is true **for all** (Figure 2) natural numbers?

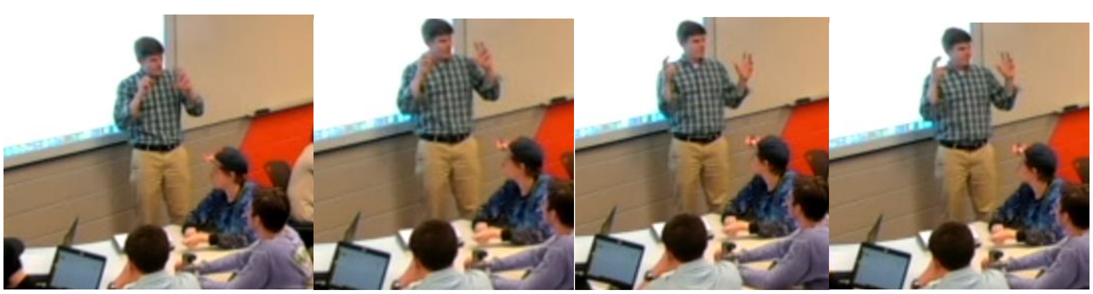

Figure 2. Example of a representational gesture used by a teacher to illustrate the quantifier "for all" ("fountain gesture").

In contrast to the "fountain" illustrating "for all" quantifier, the teacher temporarily fixed his hands motionless in the air to indicated that he fixed "some arbitrary" integer k:

Teacher: The "for all" is really important here as we just saw. So, if for all n, working for n implies working for n+1. We'd be done. Now I'm not gonna get really picky about this, but some people will want you to change the letter here. Because with n we're talking about general statement that's supposed to work for all n. And I might switch it to k for the step that [one of the students] was just talking about because I might want to say 'ok I'm just talking about some arbitrary value (Figure 3). I'm not talking about all natural numbers anymore.'

Figure 3. Example of a representational gesture used by a teacher to illustrate the quantifier "for some arbitrary" ("rigid hands").

Another purpose of using representational gestures was to attract students' attention to the logical implication between propositions P(k) and P(k+1). Norton and Arnold (2017) showed that the implication P(k)→P(k+1) may be considered as either an action that transforms P(k) into P(k+1) or as an object representing an invariant relationship between P(k) and P(k+1). Dubinsky (1991) hypothesized that treating implication as a single object is crucial for promoting students' understanding proof by induction. For this reason, the implication P(k)→P(k+1) deserved considerable attention during the observed teaching episode.

The teacher actively involved hand gesturing to facilitate the discussion on implication between P(k) and P(k+1). Typically, he used his hands in the air to track the trajectory of transition P(k) ↔ P(k+1):

Teacher: I know P(k+1) only in this [motions forward] direction. You can do an induction in both directions. You can try to show that P(k+1) **implies** P(k) (Figure 4).

Figure 4. Example of a representational gesture used by a teacher to illustrate the implication.

The students, in their turn, used similar gestures describing implication.

Teacher: All I see is 1!
Student 1: But it works for all integers.
Student 2: When k is 1 and we know 1 works then P(1+1) works (Figure 5), so 2 works.

Figure 5. Example of a representational gesture used by a student to illustrate the implication.

Metaphoric gestures

Through metaphoric gestures people put an abstract idea into a more literal, concrete form. Metaphoric gestures are similar to representational gestures in that they have a narrative character, but the images produced relate to abstract objects and processes. McNeill (1992) proposed that mathematicians have distinctive gestures for mathematical terms, and that these gestures are "somewhere on the road to a gesture language, but not all the way there" (p. 164). Thus, these are metaphoric gestures, where abstract mathematical objects are situated.

The teacher used metaphor "engine" to informally describe the idea of proof by mathematical induction:

Teacher: … So there's a lot of conjectures y'all been struggling to prove that kinda go on forever. Like that Fibonacci one where every third….umm...element of the Fibonacci sequence is even. And you start going and going and going and you can't go on forever. This [points to inductive implication] goes on forever for you. This is the **engine** that's doing all the work for you. So that's what we want to try to do. Is try to apply this **engine**.

Interestingly, a metaphoric gesture reminiscent of an engine was introduced by one of the students right before the teacher used the metaphor "engine" for the first time:

Student: Then you can plug 2 back in for k and **the logic repeats itself**.

The student revolved his hands around each other to support his idea about the repeating logic. Notably the teacher was not looking at the gesturing student (Figure 6). For this reason, it seems that the teacher *independently* came up with the gesture (Figure 7), which was employed whenever the metaphor "engine" was used.

Figure 6. One of the students is using a revolving hand. Gesturing student is put inside a white circle.

Figure7. The teacher uses hands revolving around each other to address to the metaphor "engine".

Conclusion

In line with prior research (e.g., Alibali & diRusso, 1999; Valenzeno et al., 2003; Singer & Goldin-Meadow, 2005; Abrahamson, 2003; Nemirovsky, Tierney & Wright, 1998)), the teacher and students actively used gestures in classroom discourse, and different types of gestures played different roles in mathematical conversation. This study contributes to prior research by demonstrating how different kinds of gestures played different roles in discourse on mathematical induction. Pointing gestures reduced ambiguity when the teacher was using the words "it," "this, or "that." Using this type of gesture, the instructor attempted to disambiguate the students' conflation between the truth for proposition $P(k)$ versus the implication $P(k) \rightarrow P(k+1)$. When the ambiguous word "this" was pronounced without gesturing, some of the students immediately became confused. Thus, the study confirms students' difficulty identified in the research literature about distinguishing $P(k)$ and $P(k) \rightarrow P(k+1)$ (Dubinsky, 1991; Norton & Arnold, 2017), and suggests pointing gestures as an instructional tool for ameliorating this difficulty. Further, the teacher employed numerous representing gestures to draw students' attention to the subcomponents of mathematical induction, such as quantification and logical implication. Some of these gestures were readily adopted by students. Finally, the results confirm the hypothesis that people deliberately employ metaphoric gestures when talking about abstract mathematics (McNeil, 1992; Alibali & Nathan, 2012). In this study, metaphoric gesture of "revolving hands" was independently used by teacher and students when talking about the "engine" metaphor, describing the nature of proof by mathematical induction. Future research may be conducted on comparison other instructors' gestures in teaching proof by mathematical induction.

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