Mathematical Actions, Mathematical Objects, and Mathematical Induction

Rachel Arnold
Virginia Tech

Anderson Norton
Virginia Tech

Proof by mathematical induction is arguably the most difficult proof technique for students to master. We explain this difficulty within an action-object framework. Specifically, we report on results from clinical interviews with two mathematics majors in which the first author administered tasks designed to elucidate each student's understanding of logical implications as mental objects. We found that the framework explains much of the difficulty inherent in proof by induction, even the students' struggles with hidden quantifiers.

Keywords: Action-object theory, Logical implication, Mathematical induction, Proof

Beyond the difficulties students experience with proof in general (Weber & Alcock, 2004; Mariotti, 2006; Stylianideas, 2007), mathematical induction poses particular challenges (Baker, 1996; Harel, 2002; Michaelson, 2008; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007). Proof by induction involves the implication that if a proposition holds for some integer k, then the proposition holds for the integer k+1. Students often conflate this inductive assumption with the assumption that the proof holds for any k (Avital & Libeskind, 1978; Ron & Dreyfus, 2004). Using the notation P(k) to represent the proposition applied to k, there needs to be a distinction between the implication $P(k) \rightarrow P(k+1)$ and P(k) itself.

For example, consider the proposition, P(n): The sum of the first n odd natural numbers is n^2. The proposition holds for n=1, and assuming it holds for some natural number k, we can show that it also holds for k+1. Suppose that $1 + 3 + \cdots + (2k-1) = k^2$; then adding 2k+1 (the next odd number) to both sides of the equation, we get the sum of the first k+1 natural numbers on the left side of the equation, and on the right side of the equation, we get $k^2 + 2k+1 = (k+1)^2$. Thus, we have proven that the proposition holds for n=1 and that, if P(k) is true, then P(k+1) is also true. Therefore, P(n) is true for all natural numbers.

The purpose of this paper is to investigate the cognitive origins of students' difficulties in mastering proofs by induction. More specifically, we apply an action-object theory to the logical implication $P(k) \rightarrow P(k+1)$ in order to study how a complete understanding of induction might develop. Most mathematics majors can prove logical implications (Harel & Sowder, 2007), but proof by induction imposes an additional requirement: the inductive implication, $P(k) \rightarrow P(k+1)$, must be taken as a single object rather than a relation between two objects, P(k) and P(k+1) (Dubinsky, 1986). Our study contributes empirical results to support this claim within a revised action-object framework.

We begin our report with a review of literature on students' difficulties in understanding proof in general and proof by induction. Then we introduce our action-object framework for investigating such difficulties. Next, we describe the tasks we used to investigate students' understandings within that framework. Finally, we report on results that answer the following four questions.

1. How do college mathematics majors understand logical implications?
2. Are action-object distinctions useful in modeling these understandings?
3. How do these understandings contribute to their mastery of proof by induction?

4. What other factors contribute to, or detract from, mastery of proof by induction?

Research on Students' Difficulties with Proofs and Proving

Harel and Sowder (2007) defined a conjecture as "an assertion made by an individual who is uncertain of its truth" (p. 808). Correspondingly, they defined proving as the process of removing doubts about such assertions. Recognizing the reciprocal role that conjecturing plays in proving, Boero, Lemut, and Mariotti (1996) referred to a cognitive unity between these two activities, and several researchers have described ways in which students switch back and forth between conjecturing and proving as they attempt to construct proofs (Arzarello, Andriano, Olivero, & Robutti, 1998; Herbst, 2006; Saenz-Ludlow, 1997; Weber & Alcock, 2004). For example, Cifarelli (1997) found that, "[students'] self-generated hypotheses went hand-in-hand with their conception of carrying out of purposive activity designed to test the viability of their hypotheses" (p. 20).

Harel and Sowder (2007) referred to a second kind of switching, between ascertaining and persuading. The role of persuasion in proof emphasizes its social dimension and subjective nature. In learning to make convincing arguments, students need to do more than ascertain truth for themselves; they must also find ways to convince others. Mathematical communities (such as mathematics classrooms) can specify criteria for convincing arguments. de Villiers (1999) specified six purposes these arguments might serve: verification, explanation, systemization, discovery, communication, and intellectual challenge. Additionally, mathematicians often place value on the aesthetic qualities of a proof (Sinclair, 2006).

To classify ways that students might attempt to ascertain and persuade, Harel and Sowder (2007) identified three broad proof schemes: external, empirical, and deductive. Generally, mathematics educators aspire for their students to progress toward deductive proofs because: (1) unlike external proof schemes, they include personal meaning that relates to ascertaining; and (2) unlike empirical proof schemes, they provide persuasive power via logical explanation (NCTM, 1989). Table 1 summarizes the three broad proof schemes, along with their subcategories.

In contrast to our aspirations, students generally rely on empirical or external proof schemes (Harel & Sowder, 2007), and poor performance in proving persists in college, even among mathematics majors (Selden & Selden, 2003; Weber, 2001). Proofs by mathematical induction pose particular challenges for mathematics students (and teachers), from high school through college (Avital & Libeskind, 1978; Baker, 1996; Ron & Dreyfus, 2004; Stylianides, Sylianides, & Philippou, 2007). In a conceptual analysis, Ernest (1984) speculated several possible reasons for students' difficulty, including their understandings of logical implication in general. Several empirical studies have followed, elucidating the role of such factors.

In a study of elementary and secondary school preservice teachers, Stylianides, Stylianides, and Philippou (2007) identified three specific difficulties underlying students' poor performance with proofs by mathematical induction: (1) understanding the necessity of establishing a base case (usually $n=1$); (2) interpreting the meaning of the inductive step, $P(k)$ implies $P(k+1)$; and (3) accepting that the proposition might hold beyond the cases covered by induction. In discussing the first two difficulties, the researchers cited prior work by Dubinsky (1986) suggesting that, in order to develop a mature understanding of proof by mathematical induction, students need to understand logical implication as an object: "Similar to what we found, many sophomores in Dubinsky's study tried to prove $P(k+1)$ rather than $P(k) \rightarrow P(k+1)$" (p. 162).

Table 1. *Harel and Sowder's (2007) Proof Schemes*

Class	Sub-category	Description
External Proof Schemes		
	Authoritarian	Relies on external authority, such as a text or teacher
	Ritual	Focuses on format (such as two-column proof) over substance
	Non-referential symbolic	Focuses on symbol manipulation rather than underlying concepts
Empirical Proof Schemes		
	Inductive	Relies on measurements from specific examples
	Perceptual	Relies on perceptions of specific examples
Deductive Proof Schemes		
	Transformational	Based on operational thinking and logical inference that generalizes across an entire class
	Axiomatic	A transformational proof scheme that begins from axioms.

We return to Dubinsky's work in the next section as part of a more general discussion of action-object theory. Here, we note that the conflation of the proposition P(k) with the implication P(k) → P(k+1) can lead students to conflate proofs by induction with the fallacy of assuming what is to be proved (Movshovitz-Hadar, 1993). Further, even successful mathematics students have difficulty accepting the truth of the implication without knowing the truth of the proposition itself: "How can you establish the truth of P(k+1) if you don't even know if P(k) is true?" (Avital & Libeskind, 1978, p. 430).

Harel (2002) explicitly connected students' poor performance in mathematical induction to their proof schemes, as characterized by Harel and Sowder (2007; see Table 1). In a study of preservice secondary school teachers enrolled in a college number theory course, Harel (2002) found that students' proof schemes largely fell into the empirical and external categories, particularly the authoritative and non-referential symbolic subcategories of the external proof scheme. Furthermore, he found that students' proof schemes strongly influenced the ways they understood the method of mathematical induction. He argued that students rely on authoritative schemes because they are introduced to the method before they have an intellectual need for it. He suggested a need-driven instructional approach that could build from students' empirical proof schemes toward transformational proof schemes that would support a complete understanding of the method.

The instructional approach utilizes pattern generalization, which is in the purview of the empirical proof scheme. However, the approach emphasizes patterns in the process rather than patterns in the results of that process, supporting a form of reasoning that Harel (2002) calls *quasi-induction*. In the previously shared example of summing odd integers, instruction that supports quasi-induction might involve drawing students' attention to the way one perfect square follows from the previous one, rather than the pattern of perfect squares itself. For instance, $(4+1)^2 - 4^2 = (2\cdot4+1)$, and this pattern holds across any pair of consecutive perfect squares so that

$(k+1)^2 - k^2 = (2k+1)$. Because quasi-induction and its process pattern generalizations focus on students' own mental actions rather than empirical observations, it is transformational in the Piagetian sense (Piaget, 1970), and Harel (2002) refers to it as a manifestation of the transformational proof scheme.

Action-Object Framework

Action-object theories of mathematical development derive from Piaget's (1970) genetic epistemology, in which mathematics is understood as a product of psychology: Mathematical objects arise as coordinations of mental actions through a process called reflective abstraction. Within that framework, the enterprise of mathematics education is to specify the mental actions that underlie mathematical objects and how they might be coordinated--composed and reversed--with one another to construct those objects. For example, mathematics education researchers have described how whole numbers, like 5, arise as objects for children through coordinated activities of unitizing, iterating, partitioning, and disembedding (Piaget, 1942; Steffe & Cobb, 1988; Ulrich, 2015).

Dubinsky (1986) adopted a Piagetian perspective to extend action-object theories to advanced mathematics. He developed the APOS framework to explain students' mathematical development through processes of interiorizing *actions* as *processes*, then encapsulating those processes as *objects* that can be acted upon; *schema* organize processes and objects so that students can make sense of mathematical situations. Similarly, Sfard (1991) described the reification of actions as objects, thus distinguishing objects from pseudo-objects. Unlike mathematical objects, pseudo-objects are merely figures or symbols, with no basis in action, so they cannot be de-encapsulated. For example, high school students learn rules for manipulating expressions within algebraic equations, but for many students, the expressions themselves have no reference to underlying actions (Sfard & Linchevski, 1994).

Action-object theories point to two essential features of logico-mathematical development—that students begin to construct new mathematical objects by coordinating their available mental actions and that new mental actions become available for acting on those objects. For example, students can construct the cube as a mathematical object by coordinating mental rotations, and once they have constructed the cube, they can consider new actions, like reflecting the cube about a plane through its center. The double arrow in Figure 1 represents these two essential features.

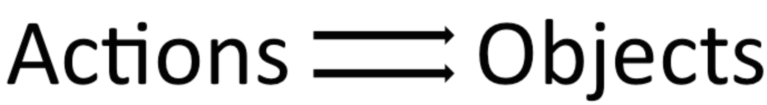

Figure 1. Actions and objects.

In the domain of proof and proving, we might consider logical implication as a mental action that transforms one assertion into another. In formal logic, this transformation is referred to as *modus ponens* (P implies Q, and P is true, therefore Q is true). It has three kinds of reverse actions: negation (P is true and Q is false); inversion, which relies on the converse of the implication (Q implies P); and *modus tollens* (P implies Q, and Q is false, therefore P is false), which relies on the contrapositive of the implication (not Q implies not P). Whereas the contrapositive of the implication is logically equivalent to the original implication, its converse is not.

"Performance on logical inferences involving modus ponens is usually reasonably good, but performance on those tasks involving modus tollens is weak, as is a full understanding of inferences involving if-then statements" (Harel & Sowder, 2007, p. 826). From a Piagetian perspective, these latter two findings go hand-in-hand. A full understanding of any mathematical object relies on the ability to reason reversibly (Piaget, 1970), so students will not have a full understanding of inferences involving if-then statements until they can reason with modus tollens as well as modus ponens. In other words, a logical implication would arise as a mathematical object for students only after they begin to coordinate modus ponens and modus tollens as reverse actions.

In referring to quasi-induction, Harel (2002) was making an action-object distinction in the development of mathematical induction. The logical implication $P(k) \rightarrow P(k+1)$ begins as an action wherein students have to carry out the transformation from the k^{th} case to then $(k+1)^{st}$ case. True induction arises from the objectification of this action (see Figure 2). "In quasi-induction one views the inference, $P(n-1) \rightarrow P(n)$, just as one of the inference steps—the last step—in a sequence of inferences that leads to $P(n)$. In mathematical induction on the other hand, one views the inference, $P(n-1) \rightarrow P(n)$, as a variable inference form, a placeholder for the entire sequence of inferences" (Harel, 2002, p. 26). Based on this action-object framework, our study focuses on the actions and objects of mathematical induction, including the two sides of the implication, the implication itself, and three ways of reversing it: converse, contrapositive, and negation.

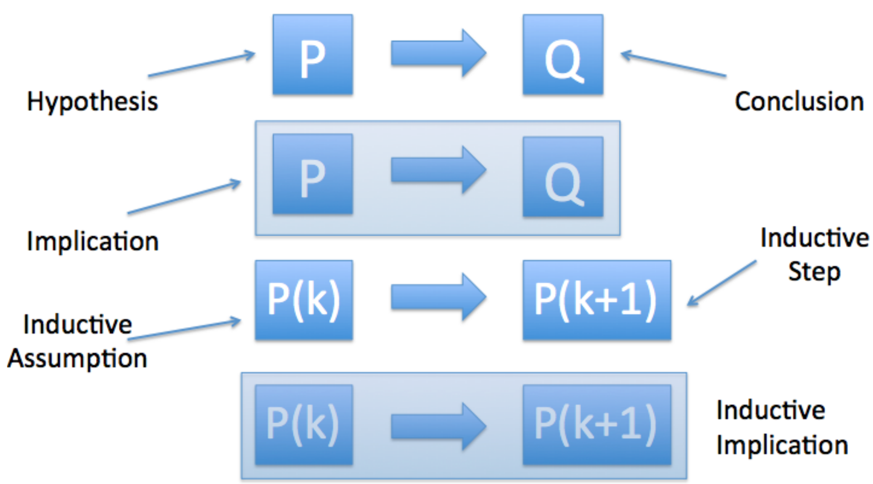

Figure 2. The actions and objects of logical implication.

Methods

To investigate the actions and objects of mathematical induction, the first author conducted clinical interviews with each of two college students, Trevor and Laura, who had completed an *Introduction to Proofs* course, which included instruction on mathematical induction. One student, Trevor, earned an A in the course, and the other student, Laura struggled in the course, earning a grade of C. In this paper, we share results from our analysis of the interview with the higher performing student, Trevor.

The one-hour interview was video-recorded and consisted of tasks designed to elicit the actions and objects those students had available for reasoning with proofs by mathematical induction. In the remainder of this section, we describe those tasks and our video analysis of the interview with Trevor.

Tasks

Interview tasks included three types (see Table 2). Type A tasks were designed to assess student understanding of logical implication. We included questions in both a familiar context (number theory) and an unfamiliar context (homology). In both contexts, students were given a statement and asked to provide truth values for its converse, its contrapositive, and its negation.

Type B tasks assessed student understanding of the components of mathematical induction (e.g., P(1) and P(k)) and how they might support an inductive proof. Type C tasks assessed student ability to construct a formal proof, both in general and by induction. Sample tasks are listed in Table 2.

Table 2. Sample interview tasks.

Task Type	**Sample Task**
A: Logical implication	Suppose the statement *S* is true. Evaluate whether the statements (a)-(c) are **true, false,** or **uncertain.** 1. *S*: *If two topological spaces are homeomorphic, their homology groups are isomorphic.* a. If two topological spaces have isomorphic homology groups, the spaces are homeomorphic. b. If the homology groups of two topological spaces are not all isomorphic, the spaces are not homeomorphic. c. There is a pair of homeomorphic topological spaces whose homology groups are not all isomorphic. 2. *S*: *Every even natural number can be written as the sum of two prime numbers.* a. If a number is not the sum of two primes, it is odd. b. There is an even number that is not the sum of two primes. c. If a number is odd, it is not the sum of two primes.
B: Induction components	Each of the following scenarios relates to a proposition P(n), where n is a positive integer. Decide whether: (a) the given information is enough to prove P(n) *without induction* (i.e., induction is not necessary); (b) the given information is enough to prove P(n) *with induction*; or (c) or the given information is *not enough* to prove the proposition. 1. P(1) is true; there is an integer $k \geq 1$ such that P(k) is true. 2. P(1) is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$. 3. P(1) is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
C: Non-inductive formal proof	Let k be an integer. Prove that if 36 divides k, then 81 divides k^2.
D: Inductive formal proof	Prove that beginning with zero, every third even number is divisible by 6.

During each interview, the first author posed tasks to the student one at a time by handing the student a slip of paper. The student was given paper to write notes and was provided opportunities to ask clarifying questions. After the student's response to each task, the interviewer would ask follow-up questions, probing the student's reasoning. For example, for Type B tasks (see Table 2), the interviewer might asked the student what additional information (s)he would need in order to show that the proposition would hold for all positive integers.

Video Analysis

Each researcher independently engaged in video analysis through an action-object lens. In this initial analysis, we focused on collecting facts about student understanding from each task type, without consideration of how performance on tasks of one type predicted performance on tasks of another type. We analyzed the students' spoken explanations for the Type A tasks to assess if they understood logical implication as an object, or if they instead viewed implication as an action between two objects, the hypothesis and the conclusion. For Type B tasks, we inferred which components of mathematical induction the students objectified and what actions they could perform on those objects. Finally, for Type C tasks, we evaluated the students' success in proving statements with and without mathematical induction.

In a second iteration of analysis, the researchers jointly considered how well the action-object framework explained student responses and how the students' performance on tasks of one type predicted their performance on later tasks. In particular, we looked for connections between the students' conceptualization of logical implication and their success in proof by mathematical induction. More specifically, we examined how the students' performance on Type A tasks predicted their objectification of the components of induction and the subsequent actions the student could perform on these components in Type B tasks. We considered whether students' objectification of logical implication and the components of mathematical induction (Type A and Type B tasks) explained additional challenges they experienced in proof by mathematical induction when compared to proof without induction (Type C tasks). In light of previous research, we considered explanations alternative to action-object theory for these differences. For example, what challenges did students experience in the components of mathematical induction because of hidden quantifiers (Shipman, 2016)? Selected transcription was used to support our analyses.

Results

We focus on the results of our video analysis of the interview with the higher-performing student, Trevor. Trevor seemed to treat logical implication as an action on two objects (P and Q), rather than a single object itself ($P \rightarrow Q$). This was evidenced throughout the interview, but particularly when he explained his reasoning for his responses to Type A tasks.

As we outline below, we infer from Trevor's spoken and written reasoning that he conceptualizes the negation, converse, and contrapositive of the implications in Task A via transformations on the objects of the implication, the hypothesis and conclusion. The more complicated the transformation process became, the more Trevor struggled with assessing the validity of the new statement.

The easiest transformation for him was by far the negation. In Task A1c, Trevor was asked to state the negation of the implication $P \rightarrow Q$ as a follow up to his response that the statement was false. Trevor replied, “the negation is P implies not Q … if you’ve already violated your assumption by saying that P is false then it doesn’t really matter what happens to Q because you don’t really care because P isn’t true.” He also wrote “P → ~Q” on his interview paper. Trevor’s explanation of why he believes the negation is P → ~Q seems to indicate the following thought process. To negate the implication, he first considers negating P and Q individually. We infer that Trevor is considering whether ~P → ~Q could be the negation. He concludes that P should not be negated, misusing the fact that an implication is vacuously true when P is false. Thus, he arrives at his conclusion that the negation should be P → ~Q. In considering the negation of the hypothesis and conclusion separately, Trevor treated implication as an action on objects, rather than an object itself. And, because Trevor believed the negation was P → ~Q, he needed only to construct ~Q as an object to assess the truth value of the entire statement.

Determining the truth value of the converse statements in Task A was slightly trickier for Trevor because it involved a reverse transformation of the implication. When considering the converse in Task A1a, Trevor began by writing out and separating the two statements in the implication (see Figure 3). He then claimed that the statement was uncertain, using the following justification: “Just because you know that the forward direction is true, there’s nothing implying that the reverse direction is true, in this case.” Prompted for a term to describe the statement in question, Trevor correctly labeled it as the converse statement. These responses indicate that Trevor treated the original statement in two parts, with implication as an transformation between them--a transformation that he could reverse. His representation of the original statement seemed to support his reasoning in comparing the reverse (converse) statement to the original.

Figure 3. Trevor’s representation of the original implication in Task 1a.

In response to Task A2c, Trevor paused for about 12 seconds, looking at the statement in question. Finally, he responded as follows:

> That one I’m going to say ‘uncertain’ because this one [pointing to the original statement] just says that if you have a naturally even number then you can express it as the sum of two primes. But it doesn’t [flips right hand over] flip the… Inverse statement isn’t true necessarily, saying that, if a number is odd, it’s not the sum of two primes. But… I feel like I’m drawing outside information into saying this next part, but being a prime number, you can’t be an even number…

Trevor went on to explain (based on his assumption that all primes are odd) that the sum of two primes will always be even. Thus, he justified the truth of the converse statement based on the context of the task. He knew that, in general, the converse would not necessarily follow from the original implication. However, once again, determining the relationship between the original statement and its converse required Trevor to perform a "flip" wherein he treated the two sides of the implication as separate objects.

While Trevor seemed to recognize the statements in tasks A1a and A2c as the converse of each implication relatively quickly, Trevor did not recognize the contrapositive statements in both tasks of Type A without significant prompting. In fact, the most challenging Type A tasks for Trevor were the contrapositive statements because they appeared to require him to combine two transformations, reversal and negation, on the objects of the implication. We note that this additional transformation, as compared to his handling of the converse, created substantially more struggle for Trevor. This is evidenced by Trevor's responses to the Task A1b, where the mathematical context was unfamiliar. After several minutes of little progress and after the first author prompting him to see the relationship between the statement *S* and its contrapositive, Trevor stated, "It's the negation of the reverse order, which is true in general. I do remember that… I don't remember the word but, 'If P then Q is true', then 'not Q implies not P' is generally true." Trevor's explanation indicates the two transformations, reversal and negation, he performs to obtain the contrapositive from *S*. Further, like in Task 1Ac, Trevor uses the term "negation" to mean the negation of each individual statement P and Q. If he was indeed referring to the "negation of the reverse order," he would have $\sim(Q \rightarrow P)$ and hence Q and ~P, which is not the contrapositive of $P \rightarrow Q$.

In Task A2a, Trevor had to again determine the truth value of the contrapositive of statement *S* in the familiar setting. Despite having just solved Task A1b, he still did not recognize the statement as the contrapositive and struggled to determine its validity. This supports our previously mentioned inference that the additional transformations necessary for Trevor to mentally construct the contrapositive are enough to blur his connection of the truth value of *S* to its contrapositive. Further, Trevor relied on his knowledge of primes to reason through his answer, ignoring the logical equivalence connection altogether. Rather than viewing the entire implication as an object that could be manipulated, he focused on the meaning of the hypothesis and conclusion separately. Consequently, we infer that Trevor's reliance on mathematical context demonstrated that he was not viewing the logical implication as a single object, rather an action on objects.

Contrasting Trevor's performance on Type A tasks in the unfamiliar versus familiar mathematical setting reveals the mental actions and objects that Trevor seemed to have available. First, because the hypothesis and conclusion of statement *S* in the unfamiliar setting (Task A1) did not carry mathematical meaning for Trevor, he seemed to conceptualize them rather quickly as pseudo-objects (Sfard, 1991). He did not devote time to framing the statements as objects with mathematical meaning. Therefore, he was more able to perform his transformations on *S* that were necessary for him to construct each new statement in Tasks A1a-c. However, in the familiar setting, Trevor got stuck trying to first construct the pieces of the implication as mental objects because they carried mathematical meaning that he believed he could unpack. As a result, he was delayed in the process of carrying out his transformations. This was particularly evidenced in Trevor's performance on Task A2a when he struggled to execute his two step action sequence of transforming the implication *S* into its contrapositive.

Next, Trevor's responses to the Type B tasks indicated that he understood how to combine components of an inductive proof, but he seemed to rely heavily on a procedure learned in class. In this way, he seemed to rely on an authoritarian proof scheme (Harel & Sowder, 2007, see Table 1). In particular, he treated induction as a sequence of objects: a base case, an inductive assumption, and an inductive step. Like his treatment of logical implication in the Type A tasks, Trevor separated the inductive implication into two objects—the assumption and the step. This reasoning was evidenced in his response to Task B2 in Table 2 where he was given P(1) and the existence of an integer k for which P(k) implies P(k+1).

> We have everything we need to do induction on this case because we have a base case that P(1) is true. And then can lump P(k) into our inductive assumption and then use P(k) implying P(k+1) to form our inductive step and follow that all the way through to all of the natural numbers.

These criteria helped him successfully distinguish which scenarios could generate a proof by induction, but a conceptual limitation became apparent. Specifically, it was clear that Trevor viewed induction as a connected sequence of three objects as follows. Trevor first checked for a base case, object one. He then linked the base case to his second object, the inductive assumption, by checking that k began at 1 for the remaining information. Once he was satisfied that this connection existed, he was able to consider the final object, the inductive step. Because of this ordered thought process, Trevor did not seem to notice that the inductive implication was missing from Task B1. In what follows, Trevor was concerned with making sure that k began at 1. He argued that if k were 3, he would still have a kind of inductive step but that the step size would be 2 instead of 1.

> We can say that P(k) is true for this given integer, but we don't really know where k is, so I don't think we can construct an inductive argument because we don't know where k is relative to 1. But if k is 3, for example, we don't know what happens at 2, and so we haven't proved it for all of the natural numbers. So even if we were to say like skip two steps, then we leave out all of the evens, for example.

We contrast this to Trevor's performance on Task B2 where he was given P(1) and that there exists an integer $k \geq 1$ such that P(k) implies P(k+1). Here, overlooking the existential quantification of k (which we address below), Trevor was satisfied with his initial action of joining the base case to the inductive assumption and completed his object sequence as follows.

> We have everything we need to do induction on this case because we have a base case that P(1) is true. And then can lump P(k) into our inductive assumption and then use P(k) implying P(k+1) to form our inductive step and follow that all the way through to all of the natural numbers.

Trevor's above response to Task B2 also surfaced a new issue in successful proof by induction: a student's ability to recognize the role of quantification in the inductive implication. Initially, Trevor overlooked the quantification of k in Tasks B1 and B2. His sequencing of inductive objects led to a correct conclusion (but for the wrong reason) that Task B1 did not have enough information, and an incorrect conclusion that Task B2 had enough information for proof by induction. However, upon seeing Task B3—which gave P(1) and that for *all* integers $k \geq 1$, P(k) implies P(k+1)—and after much prompting from the interviewer, Trevor exposed the role of quantification. He then went back to Task B2 and said there was not enough information to use induction.

While Trevor's object sequence may not lead to a mastery understanding of induction, it did allow for him to systematically outline the necessary components in a proof by induction. For the Type C task shown in Table 2, Trevor had no trouble establishing base cases, and he seemed to understand the purpose of induction.

> We can establish as many base cases as we want… 0 can be represented as something times 6; 6 can be represented as something times 6; so can 12; 18; all the way up. So you're going to have to use induction because you can't really prove every number, without sitting here and writing them all out. So you know that the inductive assumption is going to say that, assume for all numbers, k, greater than or equal to 0, but…

He struggled in establishing an inductive step and consequently completing the proof of the inductive implication: "I know what argument I want to make, but I'm not sure how to make it." His difficulty stemmed from an inability to formulate a useful representation of every third even integer. However, when prompted about what P(k+1) meant in this case, Trevor clearly understood, and also confirmed, that "k+1" did not literally mean add 1 to k. He proposed that "k+1" really meant k+6 in this case, and when asked why, he replied "because [k] is just an arbitrary number and you want to prove that the *next one* [is true]." So, Trevor did seem to understand how the inductive step should work, and arguably he could have been successful in his proof without this formulation issue.

Conclusions and Implications

Prior research identified several potential hurdles in students' mastery of proof by induction. Among these, Stylianides, Stylianides, and Philippou (2007) highlighted understanding the necessity of establishing a base case and interpreting the meaning of the inductive step. Neither student in our study demonstrated any difficulty in understanding the necessity of the base case. In fact, Trevor consistently included the base case as a critical component for inductive proofs. However, both students misconstrued the meaning of the inductive step, and the action-object framework was especially helpful in explaining why. Additionally, we uncovered a new issue not captured by prior research: students' struggles with the role of quantification in proofs by induction. We believe that our action-object framework can also be used to explain this struggle.

In line with research on proof in general (Harel & Sowder, 2007; Selden & Selden, 2003; Weber, 2001), the two students in our study relied on external proof schemes to make inductive arguments. Still, a student's ability to follow a procedural sequence of objects (base case, then inductive assumption, followed by inductive step) without a mastery of induction can allow for successful proof by induction. Separating the inductive implication into two objects, P(k) and P(k + 1), makes the process more accessible to the typical student because the typical student already handles implications in pieces (Avital & Libeskind, 1978; Dubinsky, 1986; Movshovitz-Hadar, 1993). Thus, students who have not constructed logical implications as objects can write successful proofs. However, they do not have a complete understanding for how the process works. When the inductive proof calls for some modification to the standard format (as arose in Trevor's struggles to formulate P(k + 1)) students can become confused about how to proceed.

Our results support findings from prior studies indicating that treatment of logical implications is a major mediator in students' understandings of proof by induction (Ernest, 1984; Dubinsky, 1986). Our study also affirms limitations on students' treatment of logical implications–even among high performing students like Trevor (Harel & Sowder, 2007). We consider some ways to address these limitations using our action-object framework, but first we consider the unanticipated limitation regarding students' treatment of hidden quantifiers.

Both of the students we interviewed struggled to recognize quantifiers in the statements of Type B tasks. When Trevor was asked whether there was a difference between Tasks B2 and B3 (see Table 1), he replied that they were the same. When asked if he was sure, he noticed that one sentence had the words "such that" and again overlooked the quantifiers. Other researchers have noticed students' difficulty in accounting for hidden quantifiers in proving mathematical statements (Seldon & Seldon, 1995; Shipman, 2016). In particular, Barbara Shipman discussed just how prevalent this issue of hidden quantifiers is among students and how it leads to errors in logic in proof by contradiction. The inductive implication P(k) → P(k + 1) is an implication with hidden quantifiers. P(k) and P(k + 1) are open statements that have no truth value until k is quantified. What we really mean when we write P(k) → P(k + 1) is "for all $k \geq 1$, P(k) → P(k + 1)." Students often suppress the significance of the hidden quantification of k in the inductive implication, and consequently in their conceptualization of proof by induction.

In proof by contradiction, Shipman (2016) noted that failure to recognize hidden quantifiers can lead to correct conclusions for the wrong reason, or incorrect conclusions about the validity of a statement. She also noted that oftentimes students' mistreatment of quantifiers leads to the erroneous proof of a "for all" statement by example. In our study, we found that Shipman's observations also appear to hold true in the context of proof by induction. In Task B1, Trevor came to the correct conclusion that more information was needed but for the wrong reason. He bypassed the hidden quantification of k and focused on whether P(k + 1) was true. In Task B2, Trevor's oversight of the quantification of k led to an erroneous induction proof by example. Trevor conflated showing the inductive implication was true for one k with showing the implication was true for all k. We conclude that students might be able to complete the proof of the unquantified inductive implication P(k) → P(k + 1) by breaking it down into procedural steps. However, in the absence of a memorized, quantified inductive assumption, their proofs by induction are not quite logically complete.

Students' struggles with the role of the quantification of the inductive implication in creating a logically complete proof by induction may be related to their construction of implication as a mental object. In particular, dissecting quantified statements in the context of mathematical induction places an increased cognitive demand on students that is less easily navigated if a student conceptualizes an implication as an action on objects. Because a student must work with additional objects when they are unable to mentally construct the implication as a single object, the student's cognitive resources available for addressing quantification are reduced. Our claim is supported by Trevor's performance on problems in his introduction to proofs course where the sole focus was quantification. For example, on the first exam, Trevor was given the following two statements and asked to label them as true or false and justify his answer.

1. There exists a real number x such that for all real numbers y, $2x - 3y + 7 = 14 - 6y$.
2. There exists a real number x such that for all real numbers y, $xy + 3x = 2y + 6$.

Trevor earned a perfect score on this problem, showing no problems with understanding quantifiers, even when mixed. Thus, his struggles with quantification during his clinical interview were unexpected. We speculate that Trevor's treatment of logical implication as an action on objects was an inhibiting factor.

Our study indicates that constructing logical implications as objects and identifying hidden quantifiers are prerequisite knowledge for developing transformational proof schemes for mathematical induction. Thus, similar to Harel (2002), we consider instructional activities that should be included in Proofs courses, leading into formal instruction on mathematical induction. Harel had suggested introducing quasi-induction as a means of focusing students' attention on the logical implication that related the inductive assumption to the inductive step, by explicitly relating P(k) to P(k + 1) for specific values of k. Results from our study attest to the value of that approach, assuming it supports the objectification of the implication, in general.

Tasks of Type A (see Table 1), in addition to their value in assessing whether students have constructed logical implications as objects, might also support that construction as students are challenged to transform a given logical implication into other forms (negation, converse, and contrapositive). Furthermore, our study suggests that instructors should give attention to how students handle hidden quantifiers, and tasks of Type B might reinforce the role of hidden quantifiers in proofs by induction. We recommend further study to test the efficacy of these different task types in supporting students' development of transformational schemes for proof by induction.

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